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Continuous Optimization

The multicriteria p-facility median location problem on networks

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ABSTRACT

In this paper we discuss the multicriteria p-facility median location problem on networks with positive and negative weights. We assume that the demand is located at the nodes and can be different for each criterion under consideration. The goal is to obtain the set of Pareto-optimal locations in the graph and the corresponding set of non-dominated objective values. To that end, we first characterize the linearity domains of the distance functions on the graph and compute the image of each linearity domain in the objective space. The lower envelope of a transformation of all these images then gives us the set of all non-dominated points in the objective space and its preimage corresponds to the set of all Pareto-optimal solutions on the graph. For the bicriteria 2-facility case we present a low order polynomial time algorithm. Also for the general case we propose an efficient algorithm, which is polynomial if the number of facilities and criteria is fixed.

1. Introduction

Many real-world applications are concerned with finding an optimal location for one or more new facilities on a network (road network, power grid, etc.) minimizing a function of the distances between these facilities and a given set of existing facilities (clients, demand points), where the latter typically coincide with vertices. For a recent book on location theory the reader is referred to Nickel and Puerto (2005) and references therein. Since the first seminal paper by Hakimi (1964), an ever growing number of results have been published in this field.

The majority of research focuses on the minimization of a single objective function that is increasing with distance. However, in the process of locating a new facility usually more than one decision maker is involved. This is due to the fact that often the cost incurred with the decision is relatively high. Furthermore, different decision makers may (or will) have different (conflicting) objectives. In other situations, different scenarios must be compared due to uncertainty of data or still undecided parameters of the model. One way to deal with these situations is to apply scenario analysis. Another way of reflecting uncertainty in the parameters is to consider different replications of the objective function. Hence, there exists a large number of real-world problems which can only be modeled suitably through a multicriteria approach, especially when locating public facilities.

An additional difficulty is that we are usually dealing with conflicting criteria and a single optimal solution does not always exist (which would be an optimal solution for each of the criteria). Therefore, an alternative solution concept has to be used. One possibility is to determine the set of non-dominated solutions. That is, solutions such that there exists no other solution which is at least as good for all decision makers and strictly better for at least one of them. These solutions are often called Pareto-optimal. For an overview on multicriteria location problems the reader is referred to Nickel, Puerto, and Rodríguez-Chía (2005).

In contrast to the practical needs described above, network location research involving multiple criteria has received little attention, especially when it comes to multiple facilities. In this paper, we consider the p-facility median location problem with several objective functions. Hereby, each objective function is representing the goal of one decision maker and the aim is to locate p facilities in order to minimize the total weighted distance from the clients to their closest facility. The weights assigned to clients vary from one decision maker to another, yielding different objective functions. It might even happen that one of the facilities is desirable for some decision makers and, at the same time, undesirable for others. Undesirable facilities are usually modeled using negative weights. See Eiselt and Laporte (1995) for more details on these problems. Before we discuss the literature, we present a practical example for this model. Suppose we want to locate two garbage dumps and we have a set of residential and recreational

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areas and a set of industrial sites where garbage has to be collected. There are two decision makers involved: the "Business economist" who has to keep the costs in check and the "Politician" who is concerned about the nuisance of the garbage dumps and the garbage trucks on the population. The business economist wants the dumps to be close to all sites to minimize travel times and costs. To that end he associates positive weights with the residential and industrial areas that are proportional to the average number of required garbage collections. In contrast to that, the politician wants to minimize the nuisance of the garbage dumps and of the trucks frequenting the garbage dumps for the population. Therefore, he assigns to each site a second, negative value. The smaller the weight is, the more likely it is that the residential area is far away from the dumps and the less likely it is that trucks that are not bound for these areas are simply passing through them on their way to the dumps. Formulating this problem in mathematical terms results in a bi-criteria 2-facility location model.

There are many other applications of multicriteria multifacility location problems. Bitran and Lawrence (1980) consider the multicriteria location of regional service offices in the expanding operating territories of a large property and liability insurer. These offices serve as first line administrative centers for sales support and claims processing. Another application of multiobjective optimization in the context of location theory can be found in Johnson (2001) that discusses a spatial decision making problem for housing mobility planning. Ehrgott and Rau (1999) present an analysis of a part of the distribution system of BASF SE, which involves the construction of warehouses at various locations. The authors evaluate 14 different scenarios and each of these scenarios is evaluated with the minimal cost solution obtained through linear programming and the resulting average delivery time at this particular solution. For more applications see Schöbel (2005), Carrano, Takahashi, Finseca, and Neto (2007), and Kolokolov and Zaozerskaya (2013).

Concerning the methodological aspects of multicriteria network location problems, Hamacher, Labbé, and Nickel (1999) discuss the network 1-facility problem with median objective functions. They show that for Pareto-optimal locations on undirected networks no node dominance result can be proven. Hamacher, Labbé, Nickel, and Skriver (2002) provide a polynomial time algorithm for the 1-facility problem when the objectives are both weighted median and anti-median functions. The method is generalized for any piecewise linear objective function. Zhang and Melachrinoudis (2001) develop a polynomial algorithm for the 2-criteria 1-facility network location problem maximizing the minimum weighted distance from the service facility to the nodes (maximin) and maximizing the sum of weighted distances between the service facility and the nodes (maxisum). Skriver, Andersen, and Holmberg (2004) introduce two sum objectives and criteria dependent edge lengths for the 1-facility 2-criteria problem. Nickel and Puerto (2005) solve the 1-facility problem when all objective functions are ordered medians. Colebrook and Sicilia (2007a, 2007b) provide polynomial algorithms for solving the cent-dian 1-facility location problem on networks with criteria dependent edge lengths and facilities being attractive or obnoxious.

Concerning the single criterion multifacility location problem on networks, Kalcsics (2011) derives a finite domination set for the p-median problem with positive and negative weights. For the 2-facility case, the author presents an efficient solution procedure using planar arrangements. Based on this approach, Kalcsics, Nickel, Puerto, and Rodríguez-Chía (2012) solve the 2-facility case for different equity measures.

Many of the previous papers study the problem on trees as a particular case of generalized networks. The first work dealing with several objectives and facilities is provided by Tansel, Francis, and Lowe (1982) who develop an algorithm for finding the efficient frontier of the biobjective multifacility minimax location problem on a tree network. This problem involves as objective functions the maximum of the weighted distances between specified pairs of new and existing facilities.

Despite its intrinsic interest as discussed above, to the best of our knowledge there are no papers discussing the multicriteria p-facility median location problem on networks and no results are known until the moment to obtain the set of Pareto-optimal solutions.

The remainder of this paper is organized as follows. Section 2 introduces the notation and concepts used throughout the paper. Section 3 presents some properties of the k-criteria p-facility median problem on networks. Section 4 is devoted to the development of a polynomial algorithm for the 2-criteria 2-facility version of the problem. A solution procedure for the general case is proposed in Section 5. Finally, Section 6 contains some conclusions and possible extensions of the analyzed problems.

2. Problem description and general concepts

2.1. Problem definition

Let $G = (V, E)$ be an undirected connected graph with node set $V = \{v_1, \ldots, v_n\}$ and edge set $E = \{e_1, \ldots, e_m\}$. Each edge $e \in E$ has a positive length $\ell(e)$, and is assumed to be rectifiable. Let $A(G)$ denote the continuum set of points on edges of G. We denote a point $x \in e = [u, v]$ as a pair $x = (e, t)$, where t $(0 \leqslant t \leqslant 1)$ gives the relative distance of x from node u along edge e. For the sake of readability, we identify $A(G)$ with G and $A(e)$ with e for $e \in E$. Let $k \geqslant 1$ be the number of criteria of the problem and define $Q = \{1, \ldots, k\}$. Each vertex $v_i \in V$ has a real-valued weight $w_i^q \in \mathbb{R}$, $q \in Q$. Let $J = \{1, \ldots, p\}$, where p is the number of facilities to be located. We denote by $X = (x_1, \ldots, x_p)$ the vector of locations of the facilities, where $x_j \in G$, $j \in J$. (Note that in order to allow co-location, which is quite common in location problems with negative weights, we have to represent the facility locations using a vector.) In the remainder, we use the notions location vector and solution synonymously.

We denote by $d(x, y)$ the length of the shortest path connecting two points $x, y \in G$. Let $v_i \in V$ and $x = ([v_r, v_s], t) \in G$. The distance from v_i to x entering the edge $[v_r, v_s]$ through v_r (v_s) is given as $D_i^+(x) = d(v_r, x) + d(v_r, v_i)$ ($D_i^-(x) = d(v_s, x) + d(v_s, v_i)$). Hence, the length of a shortest path from v_i to x is given by $D_i(x) = \min\{D_i^+(x), D_i^-(x)\}$. As $d(v_r, x) = t \cdot \ell(e)$ and $d(v_s, x) = (1 - t) \cdot \ell(e)$, the functions $D_i^+(x)$ and $D_i^-(x)$ are linear in x and $D_i(x)$ is piecewise linear and concave in x, cf. Drezner (1995). The distance from v_i to its closest facility is finally defined as $D_i(X) = \min_{j \in J} D_i(x_j) = \min_{j \in J}\{D_i^+(x_j), D_i^-(x_j)\}$. In the following, we call the functions $D_i^{+/-}(x)$ and $D_i(X)$ distance functions of node v_i. Moreover, we say that $D_i^a(x_j)$, $a \in \{+, -\}$, is active for X, if $D_i^a(x_j) = D_i(X)$.

We consider the objective function $F(X) = (F^1(X), \ldots, F^{|Q|}(X))$, where each $F^q(X)$, $q \in Q$, is a median function defined as:

$$F^q(X) = \sum_{i \in V} w_i^q \, D_i(X).$$

We assume the usual definition of Pareto-optimality or efficiency (Ehrgott, 2005). That is, a solution X is called efficient or Pareto-optimal, if there exists no solution X' which is at least as good as X with respect to all objective function values and strictly better for at least one value, i.e., $\nexists X' : F_q(X') \leqslant F_q(X), \ \forall q \in Q$, and $\exists q \in Q : F_q(X') < F_q(X)$. If X is Pareto-optimal, $F(X) \in \mathbb{R}^k$ will be called a non-dominated point. If $F_q(X) \leqslant F_q(X') \ \ \forall q \in Q$ and $\exists q \in Q : F_q(X) < F_q(X')$ we say X dominates X' in the decision space and $F(X)$ dominates $F(X')$ in the objective space.

The k-criteria p-facility median location problem on networks, denoted by (k,p)-MLPN, is now defined as the problem of determining the set of all Pareto-optimal solutions on the graph:

$$\operatorname*{v-min}_{X \in G \times \stackrel{p}{\ldots} \times G} F(X), \tag{1}$$

where v-min stands for vector minimization. We denote by $\tilde{X}$ the set of all Pareto-optimal solutions of (1). As mentioned in the introduction, we are interested in obtaining a description of the complete sets of Pareto-optimal solutions (in the decision space) and the non-dominated points (in the objective space). Hereby, the set of Pareto-optimal solutions comprises all alternative location vectors for the p facilities that are suitable candidates to choose from, because no other point can give rise to objective values that dominate them component-wise.

Let $h = (e_{h_1}, \ldots, e_{h_p})$ be a p-tuple of not necessarily distinct edges, where $e_{h_j} \in E, j \in J$. Then, the (k,p)-MLPN can be equivalently formulated as:

$$\text{v-min}\{F(X) | X \in e_{h_1} \times \cdots \times e_{h_p}, h \in E \times \cdots \times E\}.$$

Note that because of symmetry it is sufficient to consider only p-tuples h for which $h_1 \leqslant \cdots \leqslant h_p$.

2.2. General concepts

Let $h = (e_{h_1}, \ldots, e_{h_p})$ be a p-tuple of edges and $X \in e_{h_1} \times \cdots \times e_{h_p}$ with $x_j = (e_{h_j}, t_j), 0 \leqslant t_j \leqslant 1$. In the following, we derive a subdivision of $e_{h_1} \times \cdots \times e_{h_p}$ into maximal subsets such that the distance function of each node is linear over such a subset, i.e., each node is allocated to the same facility for all location vectors in the subset and each node reaches its closest facility via the same vertex of the edge that contains this facility. This subdivision will be a building block of our solution approach.

Let $v_i \in V$. As the functions $D_i^+(x_j)$ and $D_i^-(x_j)$ are linear for $x_j \in e_{h_j}$, the distance functions $D_i(X)$ are piecewise linear and concave for $X \in e_{h_1} \times \cdots \times e_{h_p}$. Moreover, a breakpoint of $D_i(X)$ occurs if:

- there are either two distinct facilities x_j and $x_{j'}$ at the same closest distance from v_i, i.e., $D_i(X) = D_i^a(x_j) = D_i^{a'}(x_{j'})$ for $a, a' \in \{+,-\}$, or if
- the shortest paths from v_i to its closest facility $x_j = ([v_r, v_s], t_j)$ via v_r and, respectively, v_s have the same length, i.e., $D_i(X) = D_i^+(x_j) = D_i^-(x_j)$.

It is noteworthy that the breakpoints of $D_i(X)$ for any $v_i \in V$ occur only for active functions $D_i^a(\cdot)$. See Example 1 for an illustration.

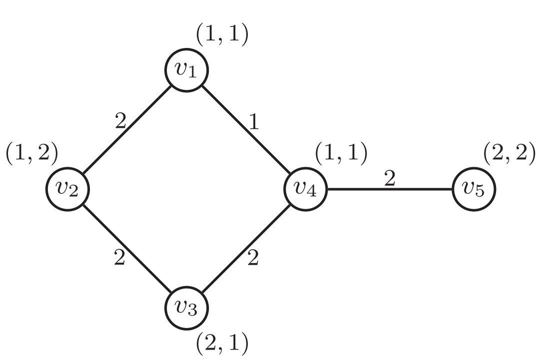

Fig. 1. Network with node weights (in brackets) and edge lengths (Example 1).

Example 1. Let $p = k = 2$ and consider the graph depicted in Fig. 1. The node weights $w_i = (w_i^1, w_i^2)$ and the edge lengths are shown in the figure.

Consider the pair of edges $h = ([v_2, v_3], [v_4, v_5])$. In Fig. 2 we depict the resulting sets of breakpoints of the distance functions over $[v_2, v_3] \times [v_4, v_5]$ (bold lines). The thin dashed lines indicate sets of intersection points between pairs of distance functions $D_i^a(\cdot)$ where at least one of the functions is not active.

The breakpoints for all other edge pairs are depicted in Appendix A.

To derive the desired subdivision, we identify each edge of the network with the unit interval $[0,1]$. Hence, the cartesian product $e_{h_1} \times \cdots \times e_{h_p}$ of the edges of h corresponds to the unit hypercube $[0,1]^p$. For the ease of notation, we identify x_j with t_j in the remainder. The sets of location vectors that fulfill the breakpoint conditions $D(X) = D_i^a(x_j) = D_i^{a'}(x_{j'})$ and $D(X) = D_i^+(x_j) = D_i^-(x_j)$ define hyperplanes in $[0,1]^p$. The set of all these hyperplanes induces a subdivision of the hypercube into subsets such that each distance function $D_i(X)$ is linear over each subset of this subdivision. Such a subdivision is also called an arrangement and the subsets are called cells, see de Berg, Cheong, van Krefeld, and Overmars (2008). As these hyperplanes resemble the breakpoints, each cell of the subdivision is maximal in the sense that all distance functions $D_i(X), v_i \in V$, are linear over the cell. As the subdivision is induced by hyperplanes, all cells are convex polygons. For more details see Kalcsics (2011). In the following, we denote by C_h the set of all cells of the subdivision for h. Moreover, for a set $D \subseteq \mathbb{R}^n, ch(D)$ denotes the convex hull of $D, ext(D)$ the set of extreme points of D, and $|D|$ the cardinality of D.

Example 1 (cont.). Fig. 2 shows the subdivision of $[0,1]^2$ into cells induced by the breakpoints for the edge pair h. In the following, we identify a solution $X = (([v_2, v_3], t_1), ([v_4, v_5], t_2))$ on the graph with the corresponding point $x = (t_1, t_2)$ on the unit square. Then, the two cells $\mathcal{C}^1$ and $\mathcal{C}^2$ of the subdivision are given by $\mathcal{C}^1 = ch(\{(0,0), (1,0), (1,1), (0.5,1), (0,0.5)\})$ and $\mathcal{C}^2 = ch(\{(0,0.5), (0.5,1), (0,1)\})$.

For any location vector X in the relative interior of a cell we either have $D_i(x_1) < D_i(x_2)$ or $D_i(x_1) > D_i(x_2)$, i.e., each node will be served by the same facility x_1 or x_2. Moreover, for each node v_i the shortest path from the node to its closest facility x_j will always pass through the same endpoint of the edge containing the facility, i.e., we either have $D_i^+(x_j) < D_i^-(x_j)$ or $D_i^+(x_j) > D_i^-(x_j)$.

3. General properties for the (k,p)-MLPN

To determine the set of Pareto-optimal solutions in the graph, we have to compute all non-dominated points of the set

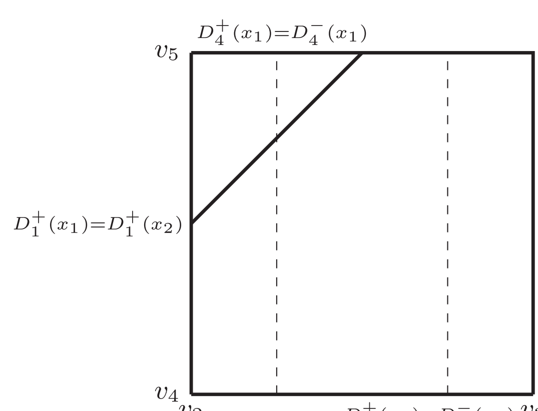

Fig. 2. Breakpoints of the distance functions $D_i(X)$ for the pair of edges $h = ([v_2, v_3], [v_4, v_5])$.

$\{F(X)|X \in G \times \overset{p}{\ldots} \times G\}$ in the objective space. To that end, using the subdivision introduced in the previous section for a given p-tuple h of edges, it will be necessary to compute in a first step the images of all cells of this subdivision. Given these images for all p-tuples h, we are then able to derive the set of non-dominated points. In a last step, we have to identify the set of location vectors on the graph whose image corresponds to the non-dominated points. These location vectors then comprise the set of Pareto-optimal solutions of our problem. In this section we discus how to compute images of cells and preimages of sets of points in the objective space. Moreover, we present some properties of the objective function of the (k,p)-MLPN. The determination of the set of non-dominated points is described in the next sections.

Let $h = (e_{h_1}, \ldots, e_{h_p})$ be a p-tuple of edges, C_h the subdivision of $[0,1]^p$ into cells, and $\mathcal{C}$ be a cell in C_h. Recall that $F(X) = (F^1(X), \ldots, F^k(X))$ is a mapping from $G \times \overset{p}{\ldots} \times G$ to $\mathbb{R}^k$. We first show how to compute images of cells.

Lemma 1 (Image of a cell). *The function F is an affine mapping over $\mathcal{C} \in C_h$, i.e., $F : \mathcal{C} \to \mathbb{R}^q$, $F(X) = At + b$, $A \in \mathbb{R}^{k \times p}$, $b \in \mathbb{R}^k$, and $t \in [0,1]^p$. Moreover, the image $F(\mathcal{C})$ of the cell has dimension $rank(A)$ with $0 \leqslant rank(A) \leqslant \min\{p,k\}$ and can be represented in the objective space as the convex hull of the images of the extreme points of $\mathcal{C}$.*

Proof. Let $X = ((e_{h_1}, t_1), \ldots, (e_{h_p}, t_p)) \in \mathcal{C}$. As each distance function $D_i(X)$ is linear over $\mathcal{C}$, so will be $F^q(X)$, $q \in Q$. Hence, we can write $F^q(X) = a_1^q t_1 + \cdots + a_p^q t_p + b^q$ where $a_j^q, b^q \in \mathbb{R}$, $q \in Q$. Therefore,

$$F(X) = \begin{pmatrix} F^1(X) \\ \vdots \\ F^k(X) \end{pmatrix} = \begin{pmatrix} a_1^1 t_1 + \ldots + a_p^1 t_p + b^1 \\ \vdots \\ a_1^k t_1 + \ldots + a_p^k t_p + b^k \end{pmatrix} = \begin{pmatrix} a_1^1 & \ldots & a_p^1 \\ \vdots & \vdots & \vdots \\ a_1^k & \ldots & a_p^k \end{pmatrix} \begin{pmatrix} t_1 \\ \vdots \\ t_p \end{pmatrix} + \begin{pmatrix} b^1 \\ \vdots \\ b^k \end{pmatrix} =: At + b$$

is an affine mapping. Moreover, $F(\mathcal{C})$ is a polytope of dimension $rank(A)$, with $0 \leqslant rank(A) \leqslant \min\{p,k\}$.

As F is an affine mapping over $\mathcal{C}$, it preserves collinearity and ratios of distances. Let $ext(\mathcal{C}) = \{v_c | c = 1, \ldots, |ext(\mathcal{C})|\}$. Since $\mathcal{C}$ is convex, $F(\mathcal{C})$ is a convex set given by

$$F(\mathcal{C}) = F(ch(\{ext(\mathcal{C})\})) = F\left(\left\{\sum_{c=1}^{|ext(\mathcal{C})|} \lambda_c v_c | \lambda_c \geqslant 0, \sum_{c=1}^{|ext(\mathcal{C})|} \lambda_c = 1\right\}\right) = \left\{\sum_{c=1}^{|ext(\mathcal{C})|} \lambda_c F(v_c) | \lambda_c \geqslant 0, \sum_{c=1}^{|ext(\mathcal{C})|} \lambda_c = 1\right\} = ch(\{F(v_c) \,|\, v_c \in ext(\mathcal{C})\}). \quad \square$$

If a proper subset $\mathcal{U}$ of the image $F(\mathcal{C})$ of a cell $\mathcal{C}$ belongs to the set of non-dominated points of $\{F(X)|X \in G \times \overset{p}{\ldots} \times G\}$ in the objective space, we have to derive the set of points of $\mathcal{C}$ whose image corresponds to $\mathcal{U}$. The next result provides a characterization of the preimage of a convex set $\mathcal{U} \subsetneq \mathcal{C}$. Its proof follows directly from the properties of affine mappings. We will see in the next sections why it is sufficient to restrict ourselves to convex sets $\mathcal{U}$.

Lemma 2 (*Preimage of a set*). *Let $\mathcal{C} \in C_h$ be a cell and $\mathcal{U}$ be a convex subset of $F(\mathcal{C})$ with extreme points $z_1, \ldots, z_\vartheta, \vartheta \geqslant 1$. The preimage $F^{-1}(\mathcal{U})$ of $\mathcal{U}$ is given by*

$$F^{-1}(\mathcal{U}) = ch(\{t \in [0,1]^p | z_c = At + b \text{ for some } c \in \{1, \ldots, \vartheta\}\}).$$

In this way, F^{-1} is well defined.

Remark 1. Note that $F^{-1}(\mathcal{U})$ depends on the cell $\mathcal{C}$. Therefore, we have to store for each point $t \in \mathbb{R}^k$ in the objective space the cell(s) who "generated" this point, i.e., to whose image $F(\mathcal{C})$ the point t belongs to.

The next example illustrates the computation of images and preimages.

Example 1 (cont.). Consider again the graph depicted in Fig. 1, and the edge pair $h = (e_{h_1} = [v_2, v_3], e_{h_2} = [v_1, v_4])$. The subdivision C_h contains a single cell that coincides with the whole unit square, i.e., $C_h = \{[0,1]^2\}$. Let $X = (x_1, x_2)$ with $x_1 = (e_{h_1}, t_1)$ and $x_2 = (e_{h_2}, t_2)$.

1. Using the weights $w_1 = (3,3)$ and $w_2 = (2,1)$ for nodes v_1 and v_2 instead of the ones depicted in Fig. 1, we obtain

$$F^1(X) = 4t_1 + 4(1-t_1) + 3t_2 + (1-t_2) + 2(2 + (1-t_2)) = 11$$
$$F^2(X) = 2t_1 + 2(1-t_1) + 3t_2 + (1-t_2) + 2(2 + (1-t_2)) = 9.$$

Hence,

$$F(X) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} + \begin{pmatrix} 11 \\ 9 \end{pmatrix}.$$

Since $rank(A) = 0$, the image $F(\mathcal{C})$ of $\mathcal{C} = [0,1]^2$ is a single point, namely $(11,9)$. Furthermore, $F^{-1}(F(\mathcal{C})) = \mathcal{C} \cap \mathbb{R}^2 = \mathcal{C}$. Fig. 3 shows $\mathcal{C}$, its image $F(\mathcal{C})$ and the preimage of $F(\mathcal{C})$.

2. Using now the alternative weights $w_1 = (3,2)$ and $w_5 = (1,2)$ for nodes v_1 and v_5, we obtain

$$F^1(X) = 2t_1 + 4(1-t_1) + 3t_2 + (1-t_2) + (2 + (1-t_2)) = 8 - 2t_1 + t_2$$
$$F^2(X) = 4t_1 + 2(1-t_1) + 2t_2 + (1-t_2) + 2(2 + (1-t_2)) = 9 + 2t_1 - t_2.$$

Hence,

$$F(X) = At + b = \begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} + \begin{pmatrix} 8 \\ 9 \end{pmatrix}.$$

Since $rank(A) = 1$, the image of $\mathcal{C}$ is now a line segment given by $ch(\{(6,11),(9,8)\})$. For computing the preimage, let $\mathcal{U} = ch(\{(8.5,8.5),(7.5,9.5)\}) \subsetneq F(\mathcal{C})$. Then,

$$F^{-1}(\mathcal{U}) = \mathcal{C} \cap ch\left(\left\{\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \middle| \begin{pmatrix} 8.5 \\ 8.5 \end{pmatrix} = At + b \text{ or } \begin{pmatrix} 7.5 \\ 9.5 \end{pmatrix} = At + b\right\}\right) = \mathcal{C} \cap ch\left(\left\{\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} | 2t_1 - t_2 + 0.5 = 0 \text{ or } 2t_1 - t_2 - 0.5 = 0\right\}\right).$$

Hence, $F^{-1}(\mathcal{U})$ is the set of all points of the square $[0,1] \times [0,1]$ between the two parallel lines defined by $2t_1 - t_2 + 0.5 = 0$ and $2t_1 - t_2 - 0.5 = 0$. Fig. 4 depicts $\mathcal{C}$, its image $F(\mathcal{C})$, and the preimage of $\mathcal{U} \subseteq F(\mathcal{C})$.

3. Using the alternative weight $w_1 = (1,2)$ for node v_1, we obtain

$$F^1(X) = 2t_1 + 4(1-t_1) + t_2 + (1-t_2) + 2(2 + (1-t_2)) = 11 - 2t_1 - 2t_2$$
$$F^2(X) = 4t_1 + 2(1-t_1) + 2t_2 + (1-t_2) + 2(2 + (1-t_2)) = 9 + 2t_1 - t_2.$$

Hence,

$$F(X) = At + b = \begin{pmatrix} -2 & -2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} + \begin{pmatrix} 11 \\ 9 \end{pmatrix}.$$

Since $rank(A) = 2$, the image of $\mathcal{C}$ is now a polygon with vertices $(7,10), (9,8), (11,9)$, and $(9,11)$. For computing the preimage, let $\mathcal{U} = ch(\{(9,11), (8,10), (9,9)\}) \subsetneq F(\mathcal{C})$. Then,

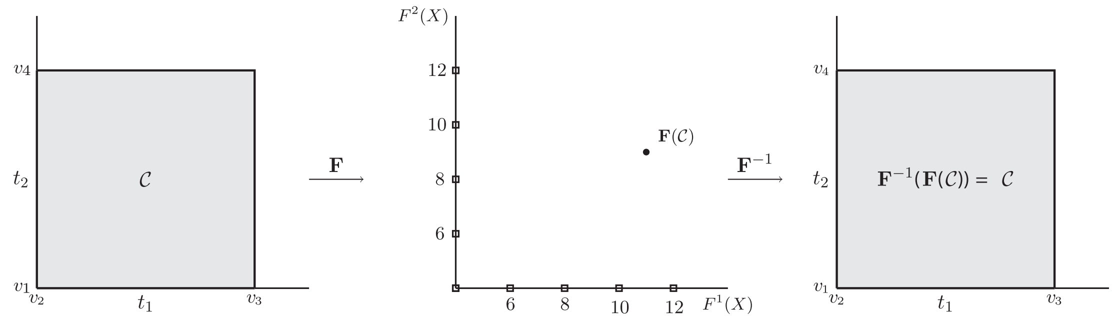

Fig. 3. The image of a cell with $rank(A) = 0$.

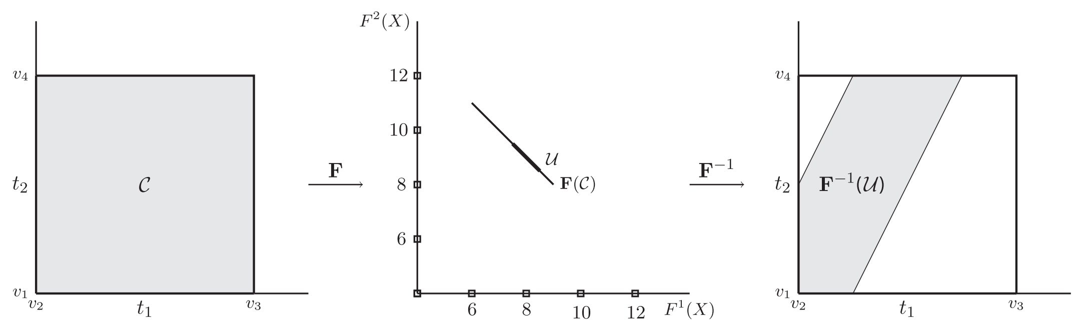

Fig. 4. The image of a cell with $rank(A) = 1$.

$$F^{-1}(\mathcal{U}) = \mathcal{C} \cap ch\left(\left\{\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \middle| \begin{pmatrix} 9 \\ 11 \end{pmatrix} = At + b, \right.\right.$$
$$\left.\left.\begin{pmatrix} 8 \\ 10 \end{pmatrix} = At + b, \text{ or } \begin{pmatrix} 9 \\ 9 \end{pmatrix} = At + b\right\}\right)$$
$$= \mathcal{C} \cap ch(\{(1,0), (5/6, 2/3), (1/3, 2/3)\}),$$

which is again a triangle. Fig. 5 shows $\mathcal{C}$, its image $F(\mathcal{C})$, and the pre-image of $\mathcal{U} \subseteq F(\mathcal{C})$.

4. A polynomial algorithm for the (2, 2)-MLPN

In this section we first discuss the (2, 2)-MLPN to explain the main ideas of our solution approach, before we turn to the general case in Section 5. In the following, we present the different steps of the approach. For a given pair of edges $h = (e_{h_1}, e_{h_2})$, we first compute the set C_h of cells of the subdivision of $[0, 1]^2$ into maximal domains of linearity of the distance functions $D_i(X)$. Afterwards, we compute the image $F(\mathcal{C})$ of each cell $\mathcal{C} \in C_h$. Depending on the rank of the mapping F with respect to $\mathcal{C}$, the image is either a point, a line segment, or a two-dimensional polygon, see Lemma 1. We store for each image a reference to the cell $\mathcal{C}$. To determine the set of non-dominated points $\widetilde{Z}$ of the images of all cells in the objective space, we adapt the approach in Hamacher et al. (1999) for the single facility bi-criteria problem. The idea of their approach is to determine the set of non-dominated points in the objective space by means of the lower envelope. To facilitate that approach, they add to the rightmost point of each image a right-open horizontal halfline (as $p = 1$, all images are points or segments). Then, each point in the objective space that is not on the lower envelope will obviously be dominated by a point on the envelope. At the end, they delete all parts of the lower envelope that belong to horizontal halflines that have been added before. The remaining points then comprise the set of all non-dominated points. Coming back to our problem, to compute the set of all non-dominated points in the objective space, we compute the image $F(\mathcal{C})$ of each cell $\mathcal{C} \in C_h$ for all edge pairs h (where it is sufficient to consider only pairs with $h_1 \leqslant h_2$). If the image is a point or a segment, we add it to a set $\mathcal{L}$ together with a reference to the respective cell. If $F(\mathcal{C})$ is a polygon, all interior points will be dominated by points on the boundary. Thus, we only add the bounding edges of the polygon to $\mathcal{L}$, again with a reference to the respective cell. Note that each polygon has at most eight bounding edges (Kalcsics, 2011). In view of the general case to be discussed in Section 5,

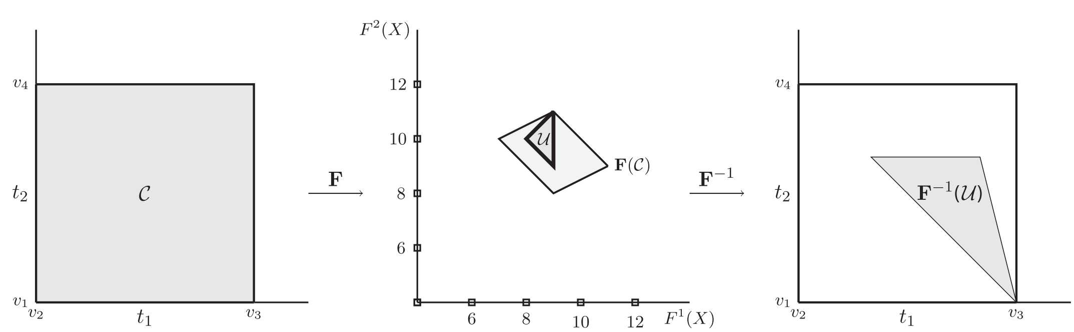

Fig. 5. The image of a cell with $rank(A) = 2$.

we generalize the approach in Hamacher et al. (1999) as follows. We add to the bottommost point of each image a horizontal and a vertical halfline extending to $+\infty$. Then, we first determine the lower envelope in y-direction and afterwards the lower envelope of the remaining points in x-direction. In this way, all dominated points will be eliminated. Formally, let le_1, le_2 denote the lower envelope functions of a set with respect to the directions of the two components of the canonical basis of $\mathbb{R}^2$. Given a collection $\mathcal{L} \subset \mathbb{R}^2$ of points and segments, Procedure 1 summarizes the steps to obtain the set of non-dominated points in the objective space.

Procedure 1. (Computing the non-dominated set of a collection $\mathcal{L} \subset \mathbb{R}^2$). <ce:para id="p0330>

1. For each connected component of $\mathcal{L}$ find the point (z_1, z_2) of the component which has the smallest z_2 value and augment the horizontal halfline $\{(z_1 + s, z_2) \mid s \in \mathbb{R}^+\}$ and the vertical halfline $\{(z_1, z_2 + s) \mid s \in \mathbb{R}^+\}$ to $\mathcal{L}$.
2. Compute $\tilde{Z} = (le_1 \circ le_2)(\mathcal{L})$.

The output $\widetilde{Z}$ of Procedure 1 is a collection of segments and points. Note that by applying both lower envelope functions, all horizontal and vertical halflines added in Step 1 are deleted at the end. Each element ℓ of $\widetilde{Z}$ is a point or a subset of a segment of $\mathcal{L}$ and contains a reference to the set of elements of $\mathcal{L}$ in which it is contained. Therefore, we can immediately determine the set $C(\ell)$ of all cells whose image contains ℓ. As, in turn, each element of $\mathcal{L}$ contains a reference to the cell generating this element, we can readily compute the preimage of ℓ with respect to each cell $\mathcal{C} \in C(\ell)$ using Lemma 2. The union of all these preimages yields the set of Pareto-optimal solutions. Algorithm 1 gives a complete description of our approach to compute the sets of non-dominated points and Pareto-optimal solutions.

Algorithm 1. Solution method for the $(2,2)$-MLPN

Input: A graph G
Output: The sets $\tilde{Z}$ and $\tilde{X}$ of non-dominated points and, respectively, Pareto-optimal solutions

1 **for** *each pair of edges* $h = (e_{h_1}, e_{h_2}) \in E \times E, h_1 \le h_2$ **do**
2   Compute the subdivision of $e_{h_1} \times e_{h_2}$ and the set C_h of all cells (Section 2.2);
3   **for** *each cell* $\mathcal{C} \in C_h$ **do**
4   Compute the linear representation of F over $\mathcal{C}$ and the image $F(\mathcal{C})$;
5   Add (the bounding edges of) $F(\mathcal{C})$ to the collection $\mathcal{L}$ and store a reference of the cell to each point/segment of the image $F(\mathcal{C})$;
6 Compute the set $\tilde{Z}$ of non-dominated points of the collection $\mathcal{L}$ using Procedure 1;
7 **for** *each* ℓ *of* $\tilde{Z}$ **do**
8   Determine $C(\ell)$;
9   **for** *each* $\mathcal{C} \in C(\ell)$ **do**
10   Add the set of points of the graph corresponding to $F^{-1}(\ell)$ to the set of Pareto-optimal solutions $\tilde{X}$;
11 **return** $\tilde{Z}$ *and* $\tilde{X}$

Complexity analysis. In the following, we discuss the complexity of Algorithm 1. For each pair of edges, there are at most $O(n^2)$ cells in C_h (Kalcsics, 2011). Using the procedure described in Kalcsics (2011), we can compute the linear representation of F over all cells $\mathcal{C} \in C_h$ in $O(n^2)$ total time. Step 5 can be computed in constant time as each cell has at most eight extreme points (Kalcsics, 2011). Since there are at most $O(m^2)$ pairs of edges, the overall complexity of Steps 1–5 is $O(n^2m^2)$.

Concerning Step 6, the set $\mathcal{L}$ has $O(n^2m^2)$ elements. Hence, we have to add $O(n^2m^2)$ horizontal and vertical lines, which can be done in $O(n^2m^2)$ time. The lower envelope can be computed in $O(n^2m^2 \log(nm))$ and contains $O(n^2m^2\alpha(n^2m^2))$ number of elements (Hershberger, 1989), where $\alpha(\cdot)$ is the inverse of the Ackerman's function. Hence, the overall complexity for Step 6 is $O(n^2m^2 \log(nm))$.

As for the computation of preimages of elements in the set of non-dominated points, Step 8 can be done in constant time since it is part of the output of the lower envelope algorithm. Step 10 can be carried out in constant time since the preimage of a point or segment has at most eight bounding segments. Thus, the overall complexity of Steps 7–10 is equal to the number of components of $\widetilde{Z}$, that is $O(n^2m^2\alpha(n^2m^2))$. With this, the overall complexity of Algorithm 1 is $O(n^2m^2 \log(nm))$.

Remark 2 (*Speed-up improvement*). If the image of a cell is a polygon it is not necessary to add all bounding segments of $F(\mathcal{C})$ to $\mathcal{L}$ since some of them will be dominated. To compute the set of all locally non-dominated bounding segments of $F(\mathcal{C})$, we first find the vertices u^1 and u^2 of $F(\mathcal{C})$ with the smallest F^1 and F^2 value, respectively. Then, starting at u^1 we add to $\mathcal{L}$ all bounding segments of $F(\mathcal{C})$ when walking from u^1 along $bd(F(\mathcal{C}))$ in clockwise direction towards u^2. Although this does not improve the worst case complexity, the actual time required to compute the lower envelope will decrease.

Example 1 (*cont.*). Let $C_h = \{\mathcal{C}^1, \mathcal{C}^2\}$ be the subdivision into cells obtained in Fig. 2 for $h = ([v_2, v_3], [v_4, v_5])$. Denoting a point $X = (([v_2, v_3], t_1), ([v_4, v_5], t_2))$ of the unit square as $X = (t_1, t_2)$, the cells $\mathcal{C}^1, \mathcal{C}^2$ can be described by $\mathcal{C}^1 = ch(\{(0,0), (1,0), (1,1), (0.5,1), (0,0.5)\})$ and $\mathcal{C}^2 = ch(\{(0,0.5), (0.5,1), (0,1)\})$. The description of $F(X)$ for $X \in [v_2, v_3] \times [v_4, v_5]$ depends on the cell under consideration and is given by:

$$F(X) = \begin{cases} \begin{pmatrix} -2 & 0 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} + \begin{pmatrix} 9 \\ 7 \end{pmatrix}, & \text{if } X = (t_1, t_2) \in \mathcal{C}^1; \\ \begin{pmatrix} 0 & -2 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} + \begin{pmatrix} 10 \\ 8 \end{pmatrix}, & \text{if } X = (t_1, t_2) \in \mathcal{C}^2. \end{cases}$$

Fig. 6 shows in dashed and dotted lines the image of $\mathcal{C}^1$ and, respectively, $\mathcal{C}^2$. The set of non-dominated points $\widetilde{Z}$ obtained by using Procedure 1 is given by:

$$\widetilde{Z} = \{(6,8)\} \cup \{(9-2t_1, 7+2t_1), 0.5 < t_1 \leqslant 1, 0 \leqslant t_2 \leqslant 1\},$$

and is depicted in Fig. 6 by the black segments and filled dots.

Computing the preimages of the set $\widetilde{Z}$ we obtain the following set of Pareto-optimal solutions:

$$\widetilde{X} = \{(v_2, v_5)\} \cup \{(([v_2, v_3], t_1), ([v_4, v_5], t_2)) : 0.5 < t_1 \leqslant 1, 0 \leqslant t_2 \leqslant 1\}$$

1. For each element $\theta \in \mathcal{L}$ compute the convex hull of the domination cones attached to each point $c \in \theta$. Then add to $\mathcal{L}$ the set of all facets of this set.
2. Compute $(le_1 \circ \ldots \circ le_k)(\mathcal{L})$.

Algorithm 2 now gives a description of the necessary steps required to compute the set of Pareto-optimal solutions and non-dominated points.

Algorithm 2. (k,p)-MLPN

Input: A graph G, and numbers p and k
Output: The sets $\tilde{Z}$ and $\tilde{X}$ of non-dominated points and, respectively, Pareto-optimal solutions

1 **for** *each p-tuple of edges* $h = (e_{h_1}, \ldots, e_{h_p}) \subset E, h_1 \leq \cdots \leq h_p$ **do**
2   Compute the set C_h of all cells of the subdivision $e_{h_1} \times \cdots \times e_{h_p}$ (see Section 2.2);
3   **for** *each cell* $\mathcal{C} \in C_h$ **do**
4   Find the linear representation of F over $\mathcal{C}$;
5   Add the (bounding facets of the) image $F(\mathcal{C})$ to the collection $\mathcal{L}$;
6 Compute the non-dominated subset $\tilde{Z}$ of the collection $\mathcal{L}$ by using Procedure 2;
7 **for** *each p-tuple of edges* $h = (e_{h_1}, \ldots, e_{h_p}) \subset E, h_1 \leq \cdots \leq h_p$ **do**
8   **for** *each* $\mathcal{C} \in C_h$ **do**
9   Compute $\ell = F(\mathcal{C}) \cap \tilde{Z}$;
10   Add the set of points of the graph corresponding to $F^{-1}(\ell)$ into the set of Pareto-optimal solutions $\tilde{X}$;
11 Return $\tilde{Z}$ and $\tilde{X}$;

Remark 3 (*The* $(2,2)$-*MLPN on trees*). If the underlying graph is a tree, the complexity of Algorithm 1 reduces by a factor of n since the number of cells of a subdivision of $[0,1]^2$ into linearity domains is at most $O(n)$, see Kalcsics (2011).

5. The $(\boldsymbol{k}, \boldsymbol{p})$-MLPN

In this section we show how to extend the previous results in order to derive an algorithmic approach to solve the general problem with p facilities and k criteria, i.e., the (k,p)-MLPN. Note first that the (k,p)-MLPN is NP-hard (if p is part of the input) because it generalizes the (1,p)-MLPN (Hakimi, 1965). Thus, there does not exist a polynomial algorithm to solve the (k,p)-MLPN (unless $\mathcal{P} = \mathcal{NP}$).

The outline of the approach for the general case is the same as for $p = k = 2$. For a given p-tuple of edges $h = (e_{h_1}, \ldots, e_{h_p})$ we first compute the set C_h of cells of the arrangement that gives us the subdivision into linearity domains (Kalcsics, 2011). For each cell $\mathcal{C} \in C_h$ we then compute the image $F(\mathcal{C})$ of $\mathcal{C}$. If $F(\mathcal{C})$ has dimension lower than k, we add it to a set $\mathcal{L}$. If $F(\mathcal{C})$ has dimension k we just add the facets of the induced polytope to $\mathcal{L}$. Moreover, we store for each element augmented to $\mathcal{L}$ a reference to its respective preimage (a cell $\mathcal{C} \in C_h$). We repeat this for all p-tuples h (where it is sufficient to consider only p-tuples with $h_1 \leqslant \ldots \leqslant h_p$). To compute the set of Pareto-optimal solutions $\widetilde{X}$, we adapt Procedure 1 to the general case with p facilities and k criteria as follows. Again we denote by le_q the lower envelope function of a set with respect to the direction of the qth component of the canonical basis of $\mathbb{R}^k$ (see Sharir (1994) for details on the lower envelope procedure).

Procedure 2. (Computing the non-dominated set $\widetilde{Z}$ of a collection $\mathcal{L} \subset \mathbb{R}^k$). Given a collection $\mathcal{L} \subset \mathbb{R}^k$ of polytopes (of dimension lower than or equal to $k-1$) the set $\widetilde{Z} \subseteq \mathcal{L}$ of non-dominated points of $\mathcal{L}$ can be obtained as follows:

Complexity analysis. In the following, we discuss the complexity of Algorithm 2. For each p-tuple of edges, there are at most $\eta = 2np^2$ hyperplanes of the type $D_i^a(x_j) = D_i^{a'}(x_{j'}), i \in V, a, a' \in \{+,-\}, j, j' \in \{1, \ldots, p\}$. Thus, there are $O(\eta^p)$ cells in C_h (Edelsbrunner, 1987). To compute the linear representation of F over a cell $\mathcal{C} \in C_h$, we pick an arbitrary vector $X \in \mathcal{C}$. For this X we first determine $D_i(X) = D_i^a(x_j), a \in \{+,-\}, j \in \{1, \ldots, p\}$ for all $i \in V$. Afterwards, we can compute $F^q(X) = \sum_{i \in V} w_i^q D_i(X) = a_1^q t_1 + \cdots + a_p^q t_p + b^q$. Both steps require in total $O(npk)$ time and this is done only once for each cell. To compute the image of a cell we determine $F(t) = At + b$ for every extreme point $t \in ext(\mathcal{C})$, where $|ext(\mathcal{C})| = O(\eta^p)$ (Edelsbrunner, 1987), and then we plot all these extreme points (in $O(\eta^p pk)$ time). Thus, computing the image of all cells in C_h can be done in $O(\eta^{2p} pk)$. Since we have to repeat this for each p-tuple of edges, the overall effort for Steps 1–5 is $O(m^p \eta^{2p} pk)$.

Concerning Step 6, for each element $\theta \in \mathcal{L}$ we have to compute the union of the domination cones attached to each point $c \in \theta$, i.e., the set $T = \{c + \mathbb{R}^k_{\geqslant} \,|\, c \in \theta\}$. As θ is convex, this set is identical to the convex hull of the domination cones attached to each extreme point of θ, i.e., $T = ch(\{c + \mathbb{R}^k_{\geqslant} \,|\, c \in ext(\theta)\})$. To compute T for each $\theta \in \mathcal{L}$, we restrict the domination cones to $[0, M]^k$ where M is sufficiently large. Then $T = ch(\{c \cup \{a | a = c + Me_q, \quad e_q = (0, \overset{q-1}{\ldots}, 0, 1, 0, \ldots, 0), \quad q \in Q\}: \quad c \in ext(\theta)\})$ is the convex hull of $O(\eta^p + k\eta^p)$ points. Since the convex hull of τ points in $\mathbb{R}^k$ can be computed in $O(\tau^{\lfloor \frac{k}{2}+1 \rfloor})$, the convex hull of the domination cones attached to the extreme points of a cell can be computed in $O((k\eta^p)^{\lfloor \frac{k}{2}+1 \rfloor})$. Next, Step 2 in Procedure 2 computes the lower envelope of the set $\mathcal{L}$. Sharir (1994) shows that the complexity of the lower envelope (in one direction) in $\mathbb{R}^k$ of δ surfaces or surface patches (all algebraic of constant degree, and bounded by algebraic surfaces of constant degree) is $O(\delta^{k+\epsilon})$ for any $\epsilon > 0$ with the constant of proportionality depending on ϵ, k, s (the maximum number of intersections among any k-tuple of the given surfaces) and on the shape and degree of the surface patches of their boundaries.

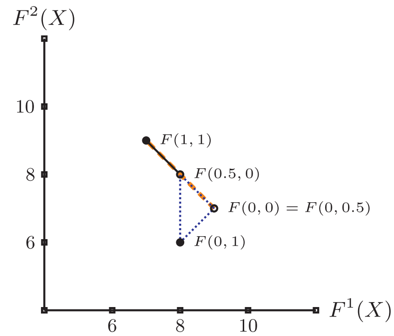

Fig. 6. Images of cells depicted in Fig. 2 (dashed and dotted lines) and non-dominated solutions (continuous line and filled dots).

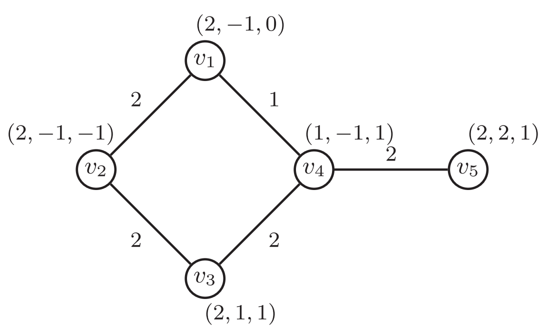

Fig. 7. Network with node weights (in brackets) and edge lengths (Example 2).

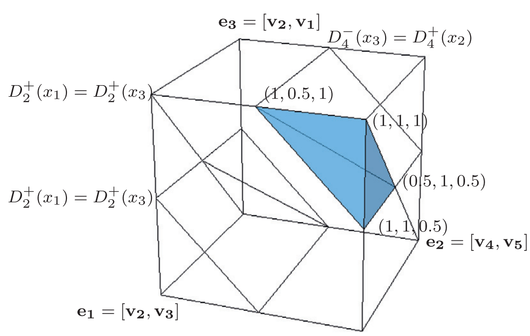

Fig. 8. Active intersection hyperplanes and subdivision into cells for edges $e_1 = [v_2, v_3]$, $e_2 = [v_4, v_5]$ and $e_3 = [v_2, v_1]$. The cell delimited by points (1,0.5,1), (1,1,1), (1,1,0.5), (0.5,1,0.5) is emphasized.

The number of facets of a polytope with τ extreme points in $\mathbb{R}^k$ is $O(\tau^{\lfloor \frac{k}{2}+1 \rfloor})$ (Gale, 1963). Thus, for each element $\theta \in \mathcal{L}$ the above convex hull construction generates at most $O((k\eta^p)^{\lfloor \frac{k}{2}+1 \rfloor})$ facets. Since each choice of a p-tuple of edges generates a subdivision with $O(\eta^p)$ cells, the input size of the lower envelope algorithm is $O(m^p\eta^p(k\eta^p)^{\lfloor \frac{k}{2}+1 \rfloor})$. Hence, the complexity for computing the lower envelope in each direction of the canonical basis is $O((m^p\eta^p(k\eta^p)^{\lfloor \frac{k}{2}+1 \rfloor})^{k+\epsilon})$ and the overall complexity is $O(k(m^p\eta^p(k\eta^p)^{\lfloor \frac{k}{2}+1 \rfloor})^{k+\epsilon})$. This implies that Step 6 can be computed in $O(k(m^p\eta^p(k\eta^p)^{\lfloor \frac{k}{2}+1 \rfloor})^{k+\epsilon})$. Note that, in the worst case, the output of the lower envelope algorithm contains as many elements as the cardinality of the input, that is, $O(m^p\eta^p(k\eta^p)^{\lfloor \frac{k}{2}+1 \rfloor})$.

Regarding Step 9, the number of facets (of dimension $p-1$) of a cell belonging to an arrangement of β hyperplanes in $\mathbb{R}^p$ (recall that there are $\eta = 2np^2$ hyperplanes of the type $D_i^a(x_j) = D_i^{a'}(x_{j'}), i \in V, a, a' \in \{+,-\}, j, j' \in \{1,\dots,p\}$) is bounded by $O(\beta)$ since each

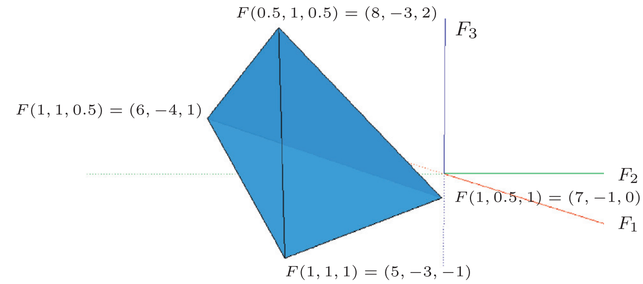

Fig. 9. Image of the cell delimited by points (1,0.5,1), (1,1,1), (1,1,0.5), (0.5,1,0.5) emphasized in Fig. 8.

hyperplane can appear at most once on each cell. Then, $\ell = F(\mathcal{C}) \cap \widetilde{Z}$ can be computed in $O(m^p\eta^{2p}(k\eta^p)^{\lfloor \frac{k}{2}+1 \rfloor})$ time, given that we have to process $O(\eta)$ facets of $F(\mathcal{C})$ with $O(m^p\eta^p(k\eta^p)^{\lfloor \frac{k}{2}+1 \rfloor})$ elements in the non-dominated set $\widetilde{Z}$. Adding one preimage of $F^{-1}(\ell)$ requires first to compute $S = \{F^{-1}(c), c \in ext(\ell)\}$ for all extreme points c of ℓ in $O(kp\eta^p)$. Later, we compute the convex hull of S in $O((kp\eta^p)^{\lfloor \frac{p}{2}+1 \rfloor})$, and intersect the result with $[0,1]^p$ in $O(2^p(kp\eta^p)^{\lfloor \frac{p}{2}+1 \rfloor})$ time. Thus, considering all cells in all p-tuples of edges, Steps 7–10 can be computed in $O(m^p\eta^p \max\{m^p\eta^{2p}(k\eta^p)^{\lfloor \frac{k}{2}+1 \rfloor}, 2^p(kp\eta^p)^{\lfloor \frac{p}{2}+1 \rfloor}\})$. Finally, the overall complexity of the complete algorithm is $O(\max\{k(m^p\eta^p(k\eta^p)^{\lfloor \frac{k}{2}+1 \rfloor})^{k+\epsilon}, m^p\eta^p 2^p (kp\eta^p)^{\lfloor \frac{p}{2}+1 \rfloor}\})$ for the case $k \geqslant 3$.

For the bicriteria problem $(2,p)$-MLPN it is of interest to analyze the complexity since the lower envelope can be computed efficiently using Procedure 1. Steps 1–5 can be computed in $O(p\eta^{2p}m^p)$ time as described in Algorithm 2. The remaining steps are done as in Algorithm 1. As $|\mathcal{L}| = O(p\eta^{2p}m^p)$, Step 6 requires $O(p\eta^{2p}m^p \log(p\eta m))$ time (recall that the complexity of the lower envelope of τ elements in $\mathbb{R}^2$ is $O(\tau log \tau)$). The difference with respect to the complexity computed in Section 4 comes from the fact that in this case we cannot exploit that the number of extreme points of each cell is bounded by a constant (eight).

As for the computation of preimages of elements in the set of non-dominated points in the objective space, Step 8 can be done in constant time since it is part of the output of the lower envelope algorithm. Adding one preimage of $F^{-1}(\ell)$ requires first to compute $S = \{F^{-1}(c), c \in ext(\ell)\}$ for every extreme point c of ℓ in $O(p\eta^p)$. Later we compute the convex hull of S in $O((p\eta^p)^{\lfloor \frac{p}{2}+1 \rfloor})$, and intersect the result with $[0,1]^p$ in $O(2^p(p\eta^p)^{\lfloor \frac{p}{2}+1 \rfloor})$. Thus, considering all cells in all p-tuples of edges, Steps 7–10 can be computed in $O(\eta^p m^p 2^p (p\eta^p)^{\frac{p}{2}})$. Finally, the overall complexity of the complete algorithm is $O(\eta^p m^p 2^p (p\eta^p)^{\frac{p}{2}})$.

Next, we illustrate Algorithm 2 with an example of a 3-facility 3-objective problem.

Example 2. Let $G = (V, E)$ be the graph depicted in Fig. 7 and let $p = k = 3$. Weights $w_i = (w_i^1, w_i^2, w_i^3)$ and edge lengths are shown in the figure. Observe that nodes v_2 and v_4 contain negative weights.

As before, we identify a solution $X = ((e_1, t_1), (e_2, t_2), (e_3, t_3))$ on the graph with the corresponding point $x = (t_1, t_2, t_3)$ of the unit cube. Choosing, for example, the edges $e_1 = [v_2, v_3], e_2 = [v_4, v_5]$ and $e_3 = [v_2, v_1]$, we obtain the following three hyperplanes: $D_2^+(x_1) = D_2^+(x_3), D_1^-(x_3) = D_1^+(x_2), D_4^-(x_3) = D_4^+(x_2)$. Fig. 8 depicts these hyperplanes and shows the resulting subdivision into linearity domains as described in Section 2.2. In this case, we obtain six cells that correspond with the different possible allocations when we place a facility on each selected edge. Consider now the cell $\mathcal{C}$ delimited by the points $(1, 0.5, 1)$, $(1, 1, 1), (1, 1, 0.5), (0.5, 1, 0.5)$ depicted in grey in Fig. 8. In this cell

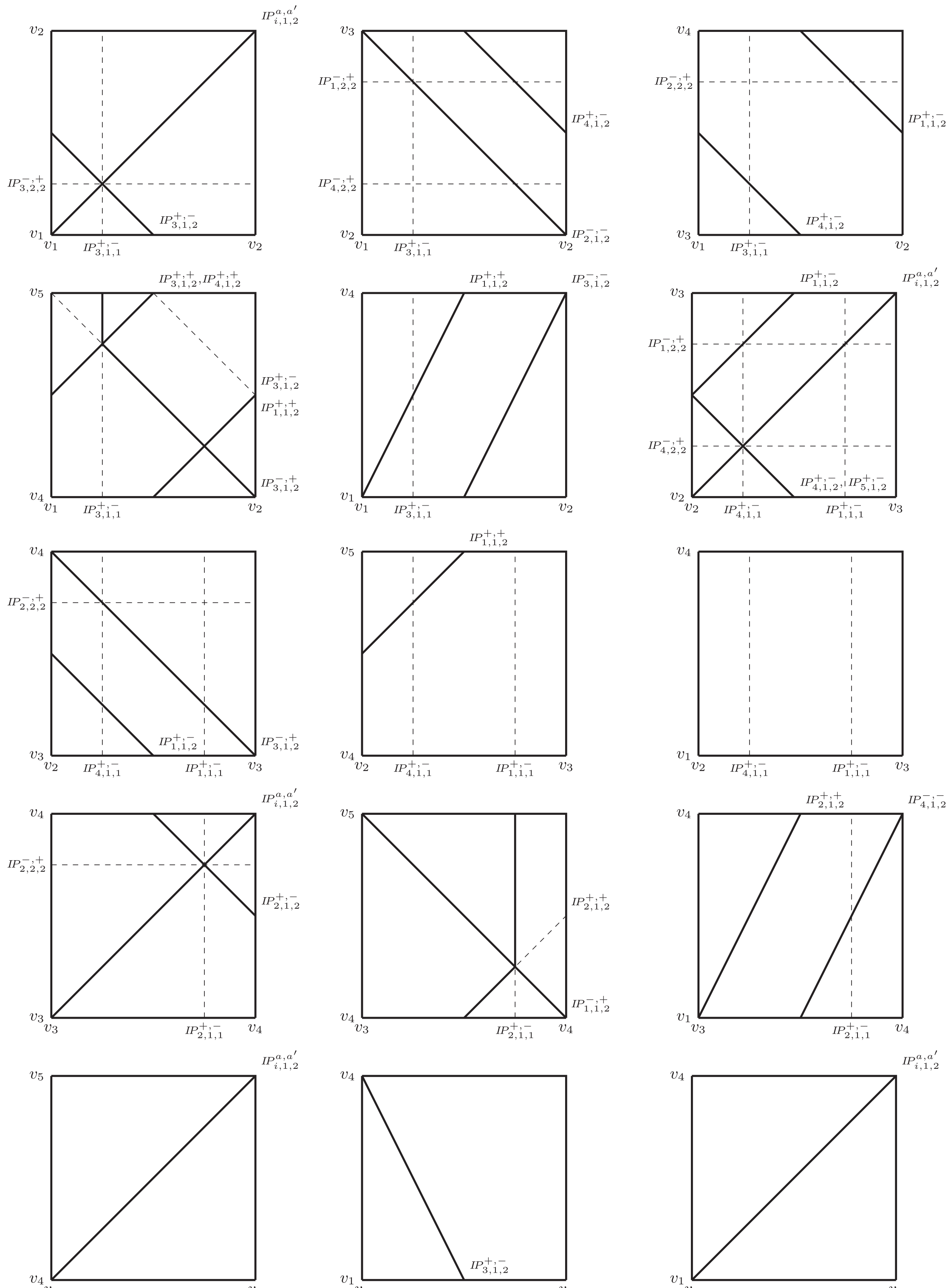

Fig. 10. Subdivision into linearity domains for all pairs of edges in Example 1. The sets of points where $D_i^a(x_j) = D_i^{a'}(x_{j'})$ for $i \in V$, $a \in \{+,-\}$, $j \in \{1,2\}$ are denoted as $IP_{ijj'}^{aa'}$.

v_1 and v_2 are allocated to the facility placed in e_3, v_3 to the one placed in e_1 and v_4, v_5 to the one placed in e_2. For this cell, the affine linear representation of F is given by

$$F(X) = \begin{pmatrix} F_1(X) \\ F_2(X) \\ F_3(X) \end{pmatrix} = \begin{pmatrix} -4 & -4 & -2 \\ -2 & -4 & +2 \\ -2 & -2 & -4 \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} + \begin{pmatrix} 15 \\ 1 \\ 7 \end{pmatrix}$$

The image of $\mathcal{C}$ in the objective space is depicted in Fig. 9.

The resulting subset of non-dominated points $\widetilde{Z}$ with respect to $\mathcal{C}$ is given by

$\widetilde{Z} = \{(7 - 2t_3, -5 + 2t_3, 3 - 4t_3), 0.5 \leqslant t_3 \leqslant 1\}$.

Computing the preimages of the set $\widetilde{Z}$, we obtain the following set of Pareto-optimal solutions with respect to $\mathcal{C}$:

$\widetilde{X} = \{(v_3, v_5, ([v_2, v_1], t_3)), 0.5 \leqslant t_3 \leqslant 1\}$

Note that two facilities are located on the vertices v_3 and v_5 and the third facility along any point of the subedge $([v_2, v_1], t_3)$, $0.5 \leqslant t_3 \leqslant 1$.

6. Conclusions

In this paper we have provided a methodology to obtain a complete description of the set of Pareto-optimal solutions for the multicriteria *p*-facility median location problem on networks. It is noteworthy that this paper is the first attempt to characterize the solution set of this problem. Note that the single criteria *p*-facility median problem is already NP-hard and handling closest assignments makes more difficult to deal with the multifacility version.

The main tools used to obtain the set of Pareto-optimal solutions is the characterization of the linearity domains of the distance functions and the lower envelope. Hence, this analysis can be easily extended to more general objective functions as long as we can again determine these domains and their image and preimage. In this sense, an open line of research is to obtain the characterization of Pareto-optimal solutions for the case of ordered median objective functions. Recall that this function includes as particular instances most classical objectives functions used in Location Theory, as for instance the median, center, *k*-center and cent-dian, see Nickel and Puerto (2005) for further details.

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Appendix A. Subdivision into linearity domains for all pairs of edges in Example 1

See Fig. 10.

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